JOURNAL OF ALGEBRA 19, 541-546 (1971)

Automorphisms of Unitary Groups

EUGENE SPIEGEL

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06268

Communicated by J. A. Dieudonné

Received December 1, 1970

1. INTRODUCTION

Let K denote a field of characteristic two endowed with an involutory automorphism. Let V denote an n-dimensional vector space over K and f a nondegenerate trace form from $V \times V$ to K. If $U_n(K, f)$ denotes the group of all isometries preserving the form f, we wish to determine the automorphisms of $U_n(K, f)$. Let $U^* = \{\alpha \in K \mid \alpha\bar{\alpha} = 1\}$. The theorem proved is:

THEOREM. *Let $n \geqslant 3$ and ψ an automorphism of $U_n(K, f)$. Then*

$$\psi(\sigma) = \chi(\sigma)\phi\sigma\phi^{-1}, \qquad \sigma \in U_n(K, f),$$

where χ is a character from $U_n(K, f)$ onto U^* and ϕ is a semilinear transformation of V onto V relative to an automorphism κ of K which permutes with $-$, and $f(\phi(x), \phi(y)) = cf(x, y)^\kappa$ identically in V for some $c \in K$.

Let ν be the Witt index of V; that is, the dimension of a maximum totally isotropic subspace of V. Dieudonné [1] questioned what the automorphisms of $U_n(K, f)$ were. In Ref. [4], the result of the theorem was shown if $\nu \geqslant 2$ and $n \geqslant 6$, while Johnson [2] recently showed that the result holds for $n \geqslant 3$ and $\nu \leqslant 1$. In order to establish the theorem it is thus sufficient to show it for $n = 4, 5$ and $\nu = 2$. It is interesting to note, that the verification of the theorem via Refs. [2, 4] and the following depends heavily on the value of ν, while the result is independent of the Witt index.

In the following we suppose the hypotheses of the theorem are satisfied.

2. CENTRALIZERS OF QUASISYMMETRIES

In order to prove the theorem, it is sufficient to show that there is an orthogonality preserving bijection of the lines of V which is induced by the automorphism ψ. Then an application of the fundamental theorem of projective geometry will give us the orthogonality preserving semilinear transformation ϕ of V, while the remaining properties of ϕ, and the determination of the character are easily established as in Ref. [1, Section 53].

If $\sigma \in U_n(K, f)$, let $C(\sigma)$ be the centralizer of σ and let $Z_n(K, f) \simeq U^*$ denote the center of $U_n(K, f)$.

For S a subset of V, let $\langle S \rangle$ denote the subspace generated by S. If x is an isotropic vector in V, let τ_x represent a transvection with respect to $\langle x \rangle$ and let $T_{\langle x \rangle}$ be the group of all transvections with respect to $\langle x \rangle$. We call the transvections τ_x and τ_y nonassociated if $\tau_y \notin T_{\langle x \rangle}$. Finally, let $[a, b]$ be the commutator of a and b.

We first state some lemmas found in Ref. [4].

LEMMA 2.1. *If $x \in V$ is isotropic, $\tau_x \in T_{\langle x \rangle}$ and $\sigma \in U_n(K, f)$, then*

$$[\tau_x, \sigma] = 1 \qquad \textit{if and only if} \qquad \sigma(\langle x \rangle) = \langle x \rangle.$$

Proof. See Ref. [4].

COROLLARY 2.2. *If $\langle x \rangle$ and $\langle y \rangle$ are distinct isotropic lines, then*

$$[\tau_x, \tau_y] = 1 \qquad \textit{if and only if} \qquad f(x, y) = 0.$$

LEMMA 2.3. *Let $x \in V$ be isotropic, $\tau_x \in T_{\langle x \rangle}$ and $\sigma \in U_n(K, f)$, then*

$$C(\sigma) = C(\tau_x) \qquad \textit{if and only if} \qquad \sigma = \mu\tau_x',$$

for some $\mu \in Z_n(K, f)$ $\tau_x' \in T_{\langle x \rangle}$.

Proof. See Ref. [4].

LEMMA 2.4. *Let $n \geq 4$, $\nu \geq 2$. If $x \in V$ is isotropic and $\tau_x \in T_{\langle x \rangle}$, then $\psi(\tau_x)$ is a transvection.*

Proof. See Ref. [4].[1]

If $x \in V$ is nonisotropic, let q_x represent a quasi-symmetry with respect to $\langle x \rangle$; that is, $q_x(y) = y$ if $y \in \langle x \rangle^\perp$, $q_x(x) = \alpha x$ for some $\alpha \in U^*$, $\alpha \neq 1$.

LEMMA 2.5. *Let $x \in V$ be nonisotropic, q_x a quasisymmetry and $\sigma \in U_n(K, f)$. Then*

$$[q_x, \sigma] = 1 \qquad \textit{if and only if} \qquad \sigma(\langle x \rangle) = \langle x \rangle.$$

[1] A group theoretic characterization of transvections appears in Yang Jun-Hui, "On the automorphisms of the Unitary Groups over fields of characteristic 2," *Acta Mathematica Sinica,* **15** (1965), 582–597.

Proof. If $[q_x, \sigma] = 1$, $q_x\sigma(x) = q_x(\sigma(x)) = \sigma q_x(x) = \sigma(\alpha x) = \alpha(\sigma(x))$ for some $\alpha \in U^*$. This says $\sigma(x)$ is an eigenvector for q_x with associated eigenvalue α. But the eigenspace for α is $\langle x\rangle$. Hence, $\sigma(x) \in \langle x\rangle$.

Conversely, if $\sigma(\langle x\rangle) = \langle x\rangle$, $\sigma(\langle x\rangle^\perp) = \langle x\rangle^\perp$.

Let $z = \gamma x + y$, $y \in \langle x\rangle^\perp$ and $\sigma q_x(z) = \sigma(\gamma\alpha x + y) = \gamma\alpha\sigma(x) + \sigma(y)$, $q_x\sigma(z) = \gamma q_x(\sigma(x)) + q_x(\sigma(y)) = \gamma\alpha\sigma(x) + \sigma(y)$ since $\sigma(y) \in \langle x\rangle^\perp$.

COROLLARY 2.6. *Let $\langle x\rangle$, $\langle y\rangle$ be distinct lines of V with $\langle x\rangle$ nonisotropic. If q_x is a quasisymmetry with respect to $\langle x\rangle$, and τ_y is either a transvection or a quasisymmetry with respect to $\langle y\rangle$, then $[q_x, \tau_y] = 1$ if and only if $f(x, y) = 0$.*

Proof. As $\langle x\rangle \neq \langle y\rangle$, $q_x(\langle y\rangle) = \langle y\rangle$ if and only if y is an eigenvector for q_x. This is the case if and only if $y \in \langle x\rangle^\perp$.

COROLLARY 2.7. *Suppose $x \in V$ is nonisotropic, q_x is a quasisymmetry with respect to $\langle x\rangle$, and $\sigma \in (U_n(K,f) - Z_n(K,f))$. If $C(\sigma) \supset C(q_x)$, then $\sigma = \mu q_x'$ with $\mu \in Z_n(K,f)$, q_x' a quasi-symmetry with respect to $\langle x\rangle$ and $C(\sigma) = C(q_x)$.*

Proof. If $y \in \langle x\rangle^\perp$, and τ_y is a quasisymmetry or transvection with respect to $\langle y\rangle$, then $f(y, x) = 0$, which implies $\tau_y \in C(q_x) \subset C(\sigma)$ so that $\sigma(\langle y\rangle) = \langle y\rangle$. Thus σ keeps all lines in $\langle x\rangle^\perp$ fixed and so there exists $\alpha \in Z_n(K,f)$ such that $\sigma\,|_{\langle x\rangle^\perp} = \alpha\,|_{\langle x\rangle^\perp}$. As $\sigma(\langle x\rangle) = \langle x\rangle$, $\alpha^{-1}\sigma$ is a quasisymmetry with respect to $\langle x\rangle$.

DEFINITION. We say $\sigma \in (U_n(K,f) - Z_n(K,f))$ is centralizer maximal if $C(\sigma) \subsetneq C(\tau)$ for $\tau \in U_n(K,f)$ implies $\tau \in Z_n(K,f)$.

Corollary 2.7 guarantees that a quasisymmetry is centralizer maximal. Transvections are also centralizer maximal.

We call $\sigma \in U_n(K,f)$ a projective quasi-symmetry with respect to $\langle x\rangle$ if σ is the product of a quasi-symmetry with respect to $\langle x\rangle$ and a center element. In view of Corollary 2.7, σ is a projective quasi-symmetry with respect to $\langle x\rangle$ if and only if $C(\sigma) = C(q_x)$ where q_x is some quasi-symmetry with respect to $\langle x\rangle$.

If we show that the image of a quasi-symmetry with respect to $\langle x\rangle$ is a projective quasi-symmetry with respect to some line $\langle y\rangle$, and let $\langle x\rangle$ correspond with $\langle y\rangle$, and as the image of a transvection with respect to, say, $\langle w\rangle$ is a transvection with respect to, say, $\langle z\rangle$, and we let $\langle w\rangle$ correspond with $\langle z\rangle$, then we will have a well defined bijective orthogonality preserving correspondence (by (2.1)–(2.6)) of the lines of V with itself. By the remark at the beginning of this section, this is sufficient to guarantee the theorem.

If $\sigma \in U_n(K,f)$ and $\alpha \in K$ is an eigenvalue of V, we call $V_\alpha(\sigma) = V_\alpha = \{x \mid \sigma(x) = \alpha x\}$ the eigenspace of σ associated to α.

3. The Case $n = 4$

Let q_x be a quasi-symmetry with respect to $\langle x\rangle$ and let $\tau = \psi(q_x)$. τ is centralizer maximal since q_x is.

As q_x commutes with a transvection, τ does also, so that there is an eigenvalue for τ. We let $V_\alpha = \{x \in V \mid \tau(x) = \alpha x\}$ be an eigenspace of τ with the property that the dimension of V_α is positive and minimal. If dim V_α is 1, and V_α is nonsingular, we let $\langle z\rangle = V_\alpha$ and q_z be a quasi-symmetry with respect to $\langle z\rangle$. But then if $\rho \in C(\tau)$, $\rho(\langle z\rangle) = \langle z\rangle$ so that $\rho \in C(q_z)$. This says $C(\tau) \subset C(q_z)$. As τ is centralizer maximal $C(\tau) = C(q_z)$ and τ is projectively a quasi-symmetry with respect to $\langle z\rangle$.

If $V_\alpha = \langle z\rangle$ and $f(z, z) = 0$, let τ_z be a transvection with respect to $\langle z\rangle$. Then $C(\tau) \subset C(\tau_z)$ and τ must be projectively a transvection. By Lemma 2.3, this is a contradiction.

The dimension of V_α is not 4 as τ is not central. Suppose $\dim(V_\alpha) = 3$. If $V_\alpha \cap V_\alpha^\perp = W$, by the minimality of the dimension of V_α, $W = \langle z\rangle$ for some isotropic vector $z \in V$. Letting τ_z be a transvection with respect to W, we have again $C(\tau) \subset C(\tau_z)$ and τ is projectively a transvection.

Finally, we may assume $\dim(V_\alpha) = 2$. If $V_\alpha^\perp = V_\alpha$, then V_α is a two-dimensional totally isotropic space, so that there exists two nonassociated commuting transvections in $C(\tau)$. But, there is only one (up to association) transvection belonging to $C(q_x)$ since $\langle x\rangle^\perp$ is a three-dimensional nonsingular space of Witt index one.

If $V_\alpha \cap V_\alpha^\perp = \langle z\rangle$, then z is isotropic and again if τ_z is a transvection with respect to $\langle z\rangle$, $C(\tau) \subset (\tau_z)$ implying that τ is projectively a transvection. We are left with the possibility that $V_\alpha \cap V_\alpha^\perp = \{0\}$. Let $\beta \in (U^* - \{\alpha\})$. Let $\sigma = \tau\,|_{V_\alpha} \oplus \beta\,|_{V_\alpha^\perp}$. If $\rho \in C(\tau)$, $\rho(V_\alpha) \subset V_\alpha$, $\rho(V_\alpha^\perp) \subset V_\alpha^\perp$ so that $C(\tau) \subset C(\sigma)$. As τ is centralizer maximal, $C(\tau) = C(\sigma)$. If $\bar\rho \in U_2(K, f\,|_{V_\alpha^\perp})$, $1\,|_{V_\alpha} \oplus \bar\rho \in C(\sigma)$ so that $\tau\,|_{V_\alpha^\perp}$ commutes with all elements of $U_2(K, f\,|_{V_\alpha^\perp})$. This says $\tau\,|_{V_\alpha^\perp} = \gamma\,|_{V_\alpha^\perp}$ for some $\gamma \in Z_n(K, f)$. q_x commutes with a transvection, so that τ does. This says that V_α or $V_\alpha^\perp$ has Witt index one. In either case, V_α and $V_\alpha^\perp$ are orthogonal hyperbolic planes. Then there exists in $C(\tau)$ two commuting nonassociated transvections. This is the desired contradiction. Hence, $\psi(\tau)$ is a projective quasi-symmetry.

4. The Case $n = 5$

Let q_x be a quasi-symmetry with respect to $\langle x\rangle$, and $\tau = \psi(q_x)$. Let V_α be an eigenspace for τ of minimum positive dimension. As in the previous section, if $\dim(V_\alpha) = 1$, τ is a projective quasi-symmetry while it is impossible that $\dim(V_\alpha) = 4, 5$.

We thus assume $\dim(V_\alpha) = 2$ or 3. In either case, if $\dim(V_\alpha \cap V_\alpha^\perp) = 1$, the centralizer of a transvection with respect to the line of $V_\alpha \cap V_\alpha^\perp$ contains $C(\tau)$ so that τ is projectively a transvection.

If $\dim(V_\alpha \cap V_\alpha^\perp) = 0$, let $\rho = \tau\,|_{V_\alpha} \oplus \beta\,|_{V_\alpha^\perp}$ where $\beta \in (U^* - \{\alpha\})$. Then $C(\tau) \subset C(\rho)$, and again by a similar argument to the corresponding case in the preceding section, we conclude that $\tau\,|_{V_\alpha^\perp} = \gamma\,|_{V_\alpha^\perp}$ for some $\gamma \in Z_n(K, f)$. Then $\dim(V_\alpha) = 2$ (by minimality) and

$$\tau = \alpha\,|_{V_\alpha} \oplus \gamma\,|_{V_\alpha^\perp}, \qquad \gamma \neq \alpha.$$

q_x commutes with a transvection, so that either V_α or $V_\alpha^\perp$ contains an isotropic vector. This implies that both V_α and $V_\alpha^\perp$ have Witt index one. As $C(q_x) \simeq C(\tau)$,

$$CC(q_x) \simeq CC(\tau).$$

But

$$CC(q_x) \simeq U^* \times U_4(K, f\,|_{\langle x\rangle^\perp})$$

while

$$CC(\tau) \simeq U_2(K, f\,|_{V_\alpha}) \times U_3(K, f\,|_{V_\alpha^\perp}).$$

Factoring out centers, we get

$$\frac{CC(q_x)}{Z(CC(q_x))} \simeq \frac{U_4(K, f\,|_{\langle x\rangle^\perp})}{Z_4(K, f\,|_{\langle x\rangle^\perp})} \simeq \frac{CC(\tau)}{Z(CC(\tau))}$$
$$\simeq \frac{U_2(K, f\,|_{V_\alpha})}{Z(K, f\,|_{V_\alpha})} \times \frac{U_3(K, f\,|_{V_\alpha^\perp})}{Z(K, f\,|_{V_\alpha^\perp})}.$$

This says that the projective unitary group $U_4(K, f\,|_{\langle x\rangle^\perp})/Z_4(K, f\,|_{\langle x\rangle^\perp})$ can be written as the direct product of two non-Abelian normal subgroups. This is impossible as any noncentral normal subgroup of $U_4(K, f\,|_{\langle x\rangle^\perp})$ contains the subgroup generated by all transvections. The transvection group is a nontrivial subgroup as the Witt index of $\langle x\rangle^\perp$ is positive.

We are thus left with the possibility that $V_\alpha \subset V_{\alpha^\perp}$ if $\dim(V_\alpha) = 2$, or $V_\alpha^\perp \subset V_\alpha$ if $\dim(V_\alpha) = 3$. In either case, we can say α is the only eigenvalue for τ, due to the minimality of the dimension of V_α.

Suppose $V_\alpha \subset V_\alpha^\perp$ and $\dim(V_\alpha) = 2$. Then there will be two nonassociated commuting transvections in $C(\tau)$, so that the Witt index of $\langle x\rangle^\perp$ is two.

Let z be a nonisotropic vector in $\langle x\rangle^\perp$ and x_1 a nonisotropic vector in $\langle x\rangle^\perp$ which is orthogonal to z and which together with z spans a hyperbolic plane H_1. Let $\langle x\rangle^\perp = H_1 \oplus H_2$. Then H_2 is hyperbolic and thus contains isotropic vectors r, s spanning it. Let $x_2 = x_1 + r$, $x_3 = x_1 + s$. If $w = x_1 + \beta z$, $\beta \in K$ is isotropic in $\langle x_1, z\rangle$, then $w + r$ is isotropic in $\langle x_2, z\rangle$

and $w+s$ is isotropic in $\langle x_3, z\rangle$. This says that $\langle x_1, z\rangle$, $\langle x_2, z\rangle$, $\langle x_3, z\rangle$ are distinct hyperbolic planes. Let q_z, q_{x_i}, $i = 1, 2, 3$, be quasi-symmetries with respect to $\langle z\rangle$, $\langle x_i\rangle$, $i = 1, 2, 3$, and $\psi(q_z) = \lambda$, $\psi(q_{x_i}) = \lambda_i$, $i = 1, 2, 3$, their images. q_z, $q_{x_i} \in C(q_x)$ so that $\lambda, \lambda_i \in C(\tau)$. τ keeps no nonisotropic lines fixed, so that λ and λ_i, $i = 1, 2, 3$, are not projective quasi-symmetries and are thus of a similar type (the remaining cases) as τ. Let V_β be a minimal eigenspace for λ and V_{β_i}, $i = 1, 2, 3$, a minimal eigenspace for λ_i.

As there is only one (up to associates) transvection in $C(q_x) \cap C(q_z)$, then $\dim(V_\alpha \cap V_\beta) = 1$. But $\langle x_1, z\rangle$, $\langle x_2, z\rangle$, $\langle x_3, z\rangle$ hyperbolic means that for $i \in \{1, 2, 3\}$ and i fixed there is a transvection in $C(q_x) \cap C(q_z) \cap C(q_{x_i})$. Hence $\dim(V_\alpha \cap V_\beta \cap V_{\alpha_i}) = 1$. Then we must have

$$\dim\left(V_\alpha \cap V_\beta \cap \left(\bigcap_{i=1}^{3} V_{\alpha_i}\right)\right) = 1.$$

This means there is a transvection in $C(\tau) \cap C(\lambda) \cap (\bigcup_{i=1}^{3} C(\lambda_i))$. But $\{x, z, x_1, x_2, x_3\}$ form a vector space basis for V so that there cannot be a transvection leaving all these lines fixed. This gives us the desired contradiction.

The case when $V_\alpha^\perp \subset V_\alpha$ and $\dim(V_\alpha) = 3$ is similar.

Therefore, we have shown that the image of q_x must be a projective quasi-symmetry and our theorem is proved.

References

1. J. Dieudonné, "On the automorphisms of the classical groups," Memoirs of the American Mathematical Society, New York, 1951.
2. A. Johnson, The automorphisms of unitary groups over a field of characteristic 2, to appear.
3. O'Meara, Group-theoretic characterization of transvections using CDC, *Math. Z.* **110** (1969), 385–394.
4. E. Spiegel, On the automorphisms of the unitary group over a field of characteristic 2, *Amer. J. Math.*, **89** (1967), 43–50.